Example on image sharpening:

Given an image $(I(x,y))_{-N\le x,y\le N}$ with DFT $(\tilde{I}(u,v))_{-N\le u,v\le N}$.

Suppose we perform unsharp masking $(H(u,v))_{-N\le u,v\le N}$ using Gaussian filter with standard deviation σ to $\tilde{I}$.

If $H(2,2)=\frac{26}{25}$. Find σ^2.

Sol:

Recall: Image sharpening = Add back high frequency components.

Given f, we aim to find $g = f + k(f - f_{smooth})$

$$DFT(g)(u,v) = \underbrace{\Big(1 + k\big(1 - H_{LP}(u,v)\big)\Big)}_{1+kH_{HP}(u,v)} DFT(f)(u,v)$$

when $k=1$, it is called unsharp masking.

For this question,

first we write down the formula of low-pass Gaussian filter:

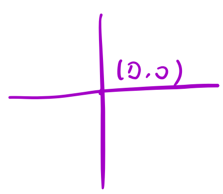
$$H_{LP}(u,v) = \exp\left(-\frac{u^2+v^2}{2\sigma^2}\right)$$

$\exp\left(-\frac{u^2+v^2}{2\sigma^2}\right)$ $\quad 1-\exp\left(-\frac{u^2+v^2}{2\sigma^2}\right)$

So, $1 + 1 - \exp\left(-\frac{4+4}{2\sigma^2}\right) = \frac{26}{25}$

$$\Rightarrow \exp\left(-\frac{8}{2\sigma^2}\right) = \frac{24}{25}$$

$$\Rightarrow \sigma^2 = \frac{4}{\log\frac{25}{24}}$$



$\exp\left(-\frac{0}{2\sigma^2}\right) = 1$

$1-\exp\left(\frac{-0}{2\sigma^2}\right) = 0$

Image denoising in the spatial domain

Example: Chapter 4 Ex 4.

4. Derive the anisotropic diffusion equation:

$$\frac{\partial I(x,y;\sigma)}{\partial \sigma} = \nabla \cdot (K(x,y)\nabla I(x,y;\sigma))$$

from minimizing the following energy functional:

$$E(I) = \int_\Omega \underbrace{K(x,y)}_{\text{given}} \|\nabla I(x,y)\|^2 \, dx\, dy.$$

We assume the continuous setting.

An grayscale image $I : \underbrace{[0,M]\times[0,N]}_{\text{spatial domain}} \to \underbrace{[0,1]}_{\text{color intensity}}$ denotes a function under certain conditions.

We then derive the PDE from the energy using calculus of variations.

Suppose f is the minimizer of the energy $E(I)$.

Then, for any function $g : [0,M]\times[0,N] \to [0,1]$,

we have $E(f+tg) \geq E(f)$.

So, $\frac{d}{dt} E(f+tg)\Big|_{t=0} \geq 0$ for any g.

$$\begin{aligned}
E(f+tg) &= \int_\Omega K(x,y) \| \nabla (f+tg)(x,y) \|^2 \, dx\, dy \\
&= \int_\Omega K(x,y) \| \nabla f(x,y) + t\nabla g(x,y) \|^2 \, dx\, dy \\
&= \int_\Omega K(x,y) \langle \nabla f(x,y) + t\nabla g(x,y), \nabla f(x,y) + t \nabla g(x,y) \rangle \, dx\, dy \\
&= \int_\Omega K(x,y) \left(\|\nabla f(x,y)\|^2 + 2t \langle \nabla f(x,y), \nabla g(x,y)\rangle + \overset{}{\underset{t^2}{}} \|\nabla g(x,y)\|^2 \right) dx\, dy
\end{aligned}$$

So, $\frac{d}{dt} E(f+tg)\Big|_{t=0}$

$$= \frac{d}{dt}\int_\Omega K(x,y)\left(\|\nabla f(x,y)\|^2 + 2t\langle \nabla f(x,y), \nabla g(x,y)\rangle + t^2\|\nabla g(x,y)\|^2\right) dx\,dy \,\Big|_{t=0}$$

$$= \int_\Omega \frac{d}{dt}\left(K(x,y)\left(\|\nabla f(x,y)\|^2 + 2t\langle \nabla f(x,y), \nabla g(x,y)\rangle + t^2\|\nabla g(x,y)\|^2\right)\right)\Big|_{t=0} dx\,dy$$

$$\frac{dt^2}{dt} = 2t$$

$$= \int_\Omega 2K(x,y)\, \nabla f(x,y)\cdot\nabla g(x,y)\, dx\,dy \geq 0 \quad (*)$$

If $(*) = \int_\Omega g\, V(f)$ for some functional $V(f)$,

then $\int_\Omega g\, V(f) \geq 0$ for any g

$\Rightarrow V(f) = 0$

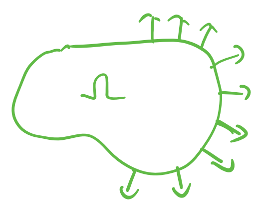
We apply divergence theorem to achieve it. $\int_\Omega \nabla\cdot\vec{F} = \int_{\partial\Omega} \vec{F}\cdot\vec{n}$

$$\int_\Omega \nabla\cdot\big(g(x,y)\,K(x,y)\nabla f(x,y)\big)\,dx\,dy = \int_{\partial\Omega} g(x,y)\,K(x,y)\nabla f(x,y)\cdot\vec{n}(x,y)\,d\sigma,$$

outward pointing normal at (x,y)

$$= \int_\Omega \nabla g(x,y)\cdot\big(K(x,y)\nabla f(x,y)\big)\,dx\,dy$$

$$+ \int_\Omega g(x,y)\,\nabla\cdot\big(K\nabla f\big)(x,y)\,dx\,dy$$



So, $\frac{1}{2}(*) = \int_{\partial\Omega} g(x,y)\,K(x,y)\nabla f(x,y)\cdot\vec{n}(x,y)\,d\sigma$

$$- \int_\Omega g(x,y)\,\nabla\cdot\big(K\nabla f\big)(x,y)\,dx\,dy \geq 0 \text{ for any } g.$$

So, $\begin{cases} K(x,y)\nabla f(x,y)\cdot\vec{n}(x,y) = 0 & \text{on } \partial\Omega \\ \nabla\cdot\big(K\nabla f\big)(x,y) = 0 & \text{on } \Omega. \end{cases}$

$$\Rightarrow \frac{\partial I(x,y;\sigma)}{\partial\sigma} = \nabla\cdot\big(K(x,y)\nabla I(x,y;\sigma)\big)$$

In the discrete case, how to solve this equation?

Starting from an initial image $f^{(0)}$,

we perform gradient descent to obtain the denoised image.

Observation:

Suppose f is not a minimizer,

we want to find some g, s.t.,

$E(f+tg) < E(f)$ for small $t>0$.

$$f \to f+tg$$

Note that $\left.\frac{d}{dt}\right|_{t=0} \frac{1}{2} E(f+tg)$

$$= \cancel{\int_{\partial\Omega} g(x,y)\, K(x,y)\, \nabla f(x,y) \cdot \vec{n}(x,y)\, d\sigma} \quad \text{ignore this term.}$$

$$- \int_{\Omega} g(x,y)\, \nabla\cdot\left(K\nabla f\right)(x,y)\, dx\, dy$$

If we choose $g(x,y) = \nabla\cdot\left(K\nabla f\right)$, $\quad -\int_{\Omega} \left(\nabla\cdot(K\nabla f)\right)^2 dx\,dy \leq 0$

then $\left.\frac{d}{dt}\right|_{t=0} \frac{1}{2} E(f+tg) < 0$

$\Rightarrow \quad E(f+tg) < E(f)$.

So, in each iteration, we need to discretize the descent direction

$\nabla\cdot\left(K\nabla f^{(k)}\right)$, then let $f^{(k+1)}(x,y) = f^{(k)}(x,y) + t\, \nabla\cdot\left(K\nabla f^{(k)}\right)(x,y)$.

We apply a finite difference scheme.

Given $f^{(k)}(u,v)$, $0 \leq u, v \leq N$.

Recall given vector field $\vec{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$,

$$\nabla\cdot\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}.$$

So, $\nabla \cdot \left(K \nabla f^{(k)} \right) (u,v)$

$$= \nabla \cdot \left(K \begin{pmatrix} \frac{\partial f^{(k)}}{\partial x} \\ \frac{\partial f^{(k)}}{\partial y} \end{pmatrix} \right)(u,v) = \frac{\partial}{\partial x}\left(K \frac{\partial f^{(k)}}{\partial x} \right)(u,v) + \frac{\partial}{\partial y}\left(K \frac{\partial f^{(k)}}{\partial y} \right)(u,v)$$

(Here we use $\frac{\partial g}{\partial x}(u,v) \approx g(u+1,v) - g(u,v)$, $\frac{\partial g}{\partial y}(u,v) \approx g(u,v+1) - g(u,v)$)

$$\approx \left(K \frac{\partial f^{(k)}}{\partial x} \right)(u+1,v) - \left(K \frac{\partial f^{(k)}}{\partial x} \right)(u,v)$$

$$+ \left(K \frac{\partial f^{(k)}}{\partial y} \right)(u,v+1) - \left(K \frac{\partial f^{(k)}}{\partial y} \right)(u,v)$$

(Here we use $\frac{\partial g}{\partial x}(u,v) \approx g(u,v) - g(u-1,v)$, $\frac{\partial g}{\partial y}(u,v) \approx g(u,v) - g(u,v-1)$)

$$\approx K(u+1,v)\left(f(u+1,v) - f(u,v) \right) - K(u,v)\left(f(u,v) - f(u-1,v) \right)$$

$$+ K(u,v+1)\left(f(u,v+1) - f(u,v) \right) - K(u,v)\left(f(u,v) - f(u,v-1) \right)$$

(Remark: you can use other methods to discretize

∇f, $\nabla \cdot (\nabla f)$, ...,

s.t., $\frac{\partial f}{\partial x}(u,v) \approx f(u+1,v) + f(u-1,v) - 2f(u,v)$

For more details and theories, take MATH 3240)